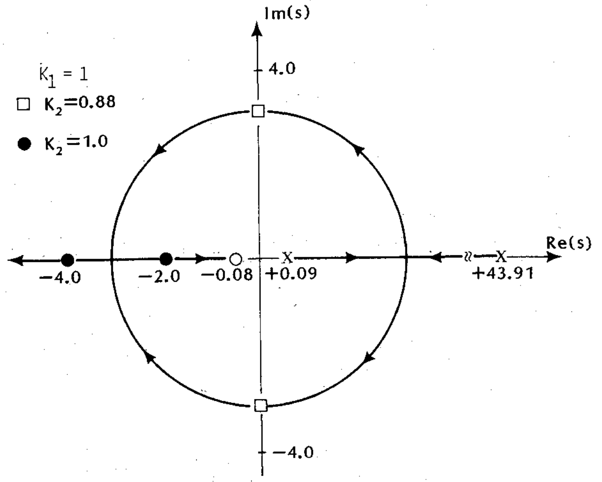

Fig. 4 Root locus vs K_2 with $K_1 = 1$ (example 2).

The coupling numerator polynomial $N_c(s) = 2$ for this example. As can be seen from Fig. 4, the second loop closure with $K_2 = 1$ brings the unstable poles back to the stable region at $s = -2$ and -4. While Ref. 3 states that this example has a $\pm\infty$dB margin from the inverse Nyquist array and characteristic loci approaches, it can be seen from Fig. 4 that it has $+\infty$dB and only -1.1 dB gain margins (imaginary axis crossing at $K_2 = 0.88$). Similarly, by closing the K_2 loop first, it can be shown that the K_1 loop has $-\infty$dB and only $+1.0$ dB gain margins. These approaches with perfect diagonalization do not consider independent gain perturbations; they consider simultaneous gain perturbations, keeping $K_1 = K_2$. The $\pm\infty$dB gain margin mentioned in Ref. 3 should not be interpreted as an independent gain variation.

Conclusions

This Note has discussed two examples that have been used for the demonstration of the superiority of multivariable control analysis/design techniques based on singular values over other design approaches. We have shown that the classical successive-loop-closure approach using simple root loci provides clear indications of lack of robustness for these examples and also gives insights into how the design can be changed so as to be more robust.

Appendix

This Appendix briefly describes a classical successive-loop-closure method[5] applied to a dynamic system with two inputs and two outputs:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{D(s)} \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{A1}$$

where s is the Laplace transform variable, y_1 and y_2 are the outputs, u_1 and u_2 are the control inputs, $D(s)$ is the characteristic determinant, and the $N_{ij}(s)$ are the numerator polynomials.

A diagonal feedback control logic is assumed:

$$u_1 = -K_1(s)y_1, \qquad u_2 = -K_2(s)y_2 \tag{A2}$$

where $K_1(s)$ and $K_2(s)$ are the diagonal feedback compensators. The closed-loop characteristic equation then becomes

$$D^2 + (K_1N_{11} + K_2N_{22})D + K_1K_2(N_{11}N_{22} - N_{12}N_{21}) = 0 \tag{A3}$$

Using the relation[5]

$$N_{11}N_{22} - N_{12}N_{21} = D(s)N_c(s) \tag{A4}$$

where $N_c(s)$ is defined as the coupling numerator polynomial, the closed-loop characteristic equation can be written as

$$D + K_1N_{11} + K_2N_{22} + K_1K_2N_c = 0 \tag{A5}$$

After the first loop closure, the characteristic equation of the second loop can be written as

$$1 + \frac{K_2(N_{22} + K_1N_c)}{D + K_1N_{11}} = 0 \tag{A6}$$

The second loop compensator, $K_2(s)$ can then be designed in a manner similar to the method for the first loop design. However, the *zeros* as well as the poles of the second loop transfer function are changed by the first loop closure. The new zeros are related to the coupling numerator. This property is useful to find the new zeros for the second loop closure in the root locus analysis. However, a detailed assessment of the multivariable robustness using the coupling numerator needs further research.

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Relationship Between Kane's Equations and the Gibbs-Appell Equations

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I. Introduction

In an earlier article in this journal Kane and Levinson[1] have shown that a certain general set of equations is particularly well suited to obtaining specific equations of motion for complex spacecraft. In subsequent books and articles Kane refers to these equations as "Kane's equations."[2] In this paper we will show that Kane's equations are simply a particular form of the Gibbs-Appell equations, which were first discovered by Gibbs[3,4] in 1879, independently discovered and studied in detail by Appell[5-8] twenty years later, and discussed in a number of textbooks.[9-14] We will do this by showing that Kane's equations are an intermediate set of equations that occurs in the derivation of the Gibbs-Appell equations from Newton's equations.

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II. Derivation of the Gibbs-Appell Equations

To simplify the presentation, we will restrict our treatment to a single particle moving in three dimensions under the action of one holonomic constraint and one anholonomic constraint. The extension of the argument to an arbitrarily complicated system acted upon by an arbitrary number of both holonomic and anholonomic constraints is straightforward.

Consider a particle of mass m, which is subject to a known force f, a holonomic constraint that reduces the configurational degrees of freedom from three to two, and an anholonomic constraint that, together with the preceding holonomic constraint, reduces the motional degrees of freedom from three to one. If we let F represent the resultant of all the constraint forces, then Newton's equations of motion for the particle are given by

$$m\ddot{x}_i = f_i + F_i, \qquad i = 1,2,3 \tag{1}$$

If we let q_1 and q_2 be any two coordinates that, together with the holonomic constraint condition, uniquely determine the position coordinates x_1, x_2, and x_3 of the particle, then the effect of the holonomic constraint on the possible positions of the particle can be described by the following relations:

$$x_i = x_i(q_1, q_2, t), \qquad i = 1,2,3 \tag{2}$$

If we now let $\dot{r}$ be any quantity linear in $\dot{q}_1$ and $\dot{q}_2$ that, together with the anholonomic constraint condition, uniquely determines the values of $\dot{q}_1$ and $\dot{q}_2$, then the effect of the anholonomic constraint on the possible motions of the particle can be described by the following relations:

$$\dot{q}_i = a_i(q_1, q_2, t)\dot{r} + b_i(q_1, q_2, t), \qquad i = 1,2 \tag{3}$$

Equations (2) and (3) define the restrictions imposed by the constraints on the position and velocity of the particle. In addition to these restrictions, we shall assume, either as included in the definitions of holonomic and anholonomic constraints or as an added requirement, that the constraint forces satisfy the following relation:

$$\sum_i F_i \frac{\partial \dot{x}_i(q_1, q_2, \dot{r}, t)}{\partial \dot{r}} = 0 \tag{4}$$

This assumption is equivalent to assuming that the constraint forces do no work in an infinitesimal virtual displacement consistent with the constraints. Equations (1–4) provide us with nine equations in the nine unknowns x_1, x_2, x_3, q_1, q_2, r, F_1, F_2, and F_3.

If we multiply Eq. (1) by

$$s_i \equiv \frac{\partial \dot{x}_i(q_1, q_2, \dot{r}, t)}{\partial \dot{r}} \equiv \frac{\partial \ddot{x}_i(q_1, q_2, \dot{r}, \ddot{r}, t)}{\partial \ddot{r}} \tag{5}$$

then sum over i and make use of Eq. (4), we obtain

$$\sum_i m\ddot{x}_i s_i = \sum_i f_i s_i \tag{6}$$

Equation (6) is Kane's equation of motion for the particle.

If we define

$$S \equiv \frac{1}{2} \sum_i m\ddot{x}_i^2 \tag{7}$$

and

$$Q \equiv \sum_i f_i s_i \tag{8}$$

then Eq. (6) can be written in the following form

$$\frac{\partial S(q_1, q_2, \dot{r}, \ddot{r}, t)}{\partial \ddot{r}} = Q \tag{9}$$

Equation (9) is the Gibbs-Appell equation of motion for the particle.

Equation (9) or (6), together with Eq. (3), provide us with three equations in the three unknowns q_1, q_2, and $\dot{r}$.

III. Discussion

Our interest in this Note is in the relationship between Eqs. (6) and (9). Are they inherently independent equations, different but equal forms of the same equation, or is one equation a limited or particular form of the other? Do they lead to the same or different results? Is the labor involved in applying one equation to a specific problem greater than that in applying the other?

Kane obviously considers Eq. (6) as an inherently independent equation. He has named it. He has constructed an elaborate notation to exploit it. He has developed a specialized terminology associated with it. Thus, $\dot{r}$ is called a generalized speed; the vector whose components are s_i is called a partial rate of change; the quantity $\Sigma_i f_i s_i$ is called a generalized active force; and the negative of the quantity $\Sigma_i m\ddot{x}_i s_i$ is called a generalized inertia force.

Most historical derivations of the Gibbs-Appell equation, Eq. (9), contain, either explicitly or implicitly, Kane's equation, Eq. (6). Hence, by implication Kane's equation is considered to be of less importance than the Gibbs-Appell equation. There are several conspicuous reasons for this. First, the Gibbs-Appell equation is more manifestly a generalized equation than Kane's equation since the function $S(q_1, q_2, \dot{r}, \ddot{r}, t)$ does not explicitly depend on the Cartesian coordinates x_1, x_2, and x_3; and the generalized force Q, rather than being defined by Eq. (8), could have been defined without reference to the Cartesian components of the force f by noting that the work done by the force in an infinitesimal virtual displacement consistent with the constraints is given by $\boldsymbol{R}\delta r$. Secondly, if the number of motional degrees of freedom is greater than one, then the Gibbs-Appell equations are generated quite simply and elegantly from a single scalar quantity S, whereas vector quantities are required to generate Kane's equations.

The pertinent question, however, is not which equation is theoretically more elegant but which equation is operationally superior. Both equations, when applied to the same system, will yield the same equation of motion for a given set of coordinates; hence, the crucial question in judging between the two methods is how much labor is required in each case to pass from the general equation of motion to the particular equation of motion. In the implementation of the Gibbs-Appell method, one obtains first the function $S(q_1, q_2, \dot{r}, \ddot{r}, t)$ and then the partial derivative $\partial S/\partial \ddot{r}$. If, in constructing S, one leaves it in the form $\frac{1}{2}\Sigma_i m_i[\ddot{x}_i(q_1, q_2, \dot{r}, \ddot{r}, t)]^2$, then, to obtain the partial derivative, one has the option of either completing the square and then taking the partial derivative or, alternatively, of leaving S in the above form, using implicit differentiation, and obtaining the partial derivative in the form $\Sigma_i m_i \ddot{x}_i[\partial \ddot{x}_i(q_1, q_2, \dot{r}, \ddot{r}, t)/\partial \ddot{r}]$, which is just the negative of Kane's generalized inertia force. With the latter option, which is always available, one thus makes immediate contact with Kane's method. Hence, in principle, there can be no problem in which the labor involved starting with the Gibbs-Appell equations significantly exceeds the labor involved starting with Kane's equation. The converse of the above is not necessarily true. There may be situations in which it is preferable to complete the squares in S or to reformulate S in some way before taking the partial derivative. In such cases, the introduction of the function S is definitely advantageous. The situation is analogous to that which exists for forces derivable from a potential. It is not necessary to introduce the potential function, but it can certainly be helpful at times.

The preceding statements are not confined to the simple system we have been considering but can easily be generalized to more complicated systems. Kane has himself provided us with an example that illustrates this. In Ref. 1 he applies both his method and the Gibbs-Appell method to a very complex problem, and on the basis of his results concludes that the labor involved in applying the Gibbs-Appell equation greatly exceeds the labor involved in applying his equations. However, in his calculations Kane failed to take advantage of the option described in the preceding paragraph. In the sentence immediately following Eq. (93) in Ref. 1, where, in the application of the Gibbs-Appell equations, he is faced with the necessity of taking the derivative of the square of a very complicated term, he assumes that it is necessary first to complete the square and then to take the derivative. Had he at this point implicitly differentiated, he would have ended up having to carry out exactly the same operations as required in the application of his own equations and would consequently have demonstrated that the labor involved starting with the Gibbs-Appell equations was the same as the labor involved starting with his equations. Since it was the results in Ref. 1 that ultimately led Kane to use the term "Kane's equations,"[2] one wonders what would have happened had he not, on the basis of this unfortunate oversight, erroneously concluded that the Gibbs-Appell method was more laborious than his own method.

It should be noted in our derivation of the Gibbs-Appell equation that none of the arguments depended on using the concept of virtual displacements or virtual work. Although it is not necessary to use these concepts in deriving or applying either Kane's equation or the Gibbs-Appell equation, they are nevertheless at times very helpful.

IV. Conclusion

From the preceding arguments, we conclude that Kane's equations are simply a particular form of the Gibbs-Appell equations and that Kane's method is simply a particular method of applying the Gibbs-Appell equations.

Although Kane cannot be credited with creating Kane's equations, he is responsible for resurrecting them, exploiting them, and promoting their use.

Concerning terminology, we suggest that Eq. (6) be called either Kane's form of the Gibbs-Appell equation or the Gibbs-Appell-Kane equation. We also suggest that Eq. (9) be called the Gibbs-Appell equation, which is the term, following Pars,[12] we have adopted. Most authors refer to Eq. (9) as Appell's equation. Kane, however, contends that Eq. (9) is "erroneously attributed to Appell"[1] and calls it Gibbs' equation.

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Rotation of a Triaxial Satellite near the Lagrangian Point L_4

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Introduction

IN recent years, the problem of the motion of a rigid body placed in the libration points of the restricted three-body problem has attracted the attention of several authors in relation to the possible use of such points in space flights. In those papers, one of the basic assumptions is that the center of the masses of the satellite remains exactly at the libration points.

This problem was started by Kane and Marsh,[1] who considered an axisymmetric satellite whose axis of symmetry is perpendicular to the orbital plane and made a study of the stability of the motion. This work was continued by Robinson,[2-4] who obtained the attitude stability for a triaxial satellite. Barkin[5] obtained stationary solutions to the rotation of the satellite placed at the Lagrangian point L_4 of the Earth-moon system by integrating the Eulerian equations of motion. Also, he found some families of periodic orbits, together with the conditions for their stability. Barkin and Markov[6] presented the equations of motion in terms of Delaunay-Hill's averaging scheme for the resonant case. An analogous study was made by Sidlichovsky,[7] who presented the solution of the first-order equations of the Lie-Hori theory. Gamarnik[8] and Krasilnikov[9] made some studies supporting the thesis that the center of masses moves along an arbitrary periodic orbit near the point L_4. Markeev[10] made an analogous study for the collinear point L_2. Elipe and Ferrer[11] analyzed the nonresonant case of an axisymmetric rigid body by means of the Lie-Deprit perturbation method, when the center of mass O_3 is moving in a neighborhood of L_4. In the resonant case, it is shown that the Hamiltonian is reduced to a generalized ideal resonance problem.

In the present Note, we give a first-order solution for the triaxial satellite, applying the Cid et al.[12] solution for the orbital motion. The solution for the axisymmetric rigid body mentioned above is obtained here as a particular case.

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